An Economic Approach to Optimize Capital Allocation

Abstract

A central problem that investors face is how to manage risk on transactions with positive expectation of profit while maximizing the rate of growth of capital. One way to define the problem is given a transaction with a positive expectation of profit, how much risk should the investor take on the transaction to maximize long term growth of its capital base by maximizing risk adjusted return while minimizing ruin. In this paper we explore the use of Kelly Criterion, which is to maximize the expected value of the logarithm of Insurers capital ("maximize expected logarithmic utility") to find solutions to the problem. The criterion is known to economists and financial theorists by names such as the "geometric mean maximizing portfolio strategy", maximizing logarithmic utility, the growth-optimal strategy, including the capital growth criterion. We prove these concepts using a hypothetical investor with a fixed probability of total loss. This approach can be generally utilized in allocating capital for investments.

Keywords: Kelly criterion, Betting, Long run investing, Portfolio allocation, Logarithmic utility, Capital growth

Introduction

The problem in investing is to find risks with prices that produce excess risk-adjusted expected rates of return. Once these favorable opportunities have been identified, the investor must decide how much of that risk they should take, how much of their capital they should put at risk to maximize their risk adjusted return thus maximizing their capital base while capping their downside. We can study this problem by creating a framework to value capital using a utility function. These utility functions are non-decreasing (more Capital is at least as good as less Capital). Some examples are as follows:

$U(X) = X^a,\ 0 <= a < \infty$, *and or*

$U(X) = Ln\ X$, *where* $\ln 0 = -\infty$,

where X = InvestorCapital

J. L. Kelly utilized the logarithmic Utility function described above. Kelly criterion chooses to maximize the expected growth rate function. Which, once a utility function of Capital is specified, boils down to maximizing the expected value of the Utility of Capital. Kelly tries to maximize $E[Ln(X)]$, the expected value of the logarithm of the (random variable). In our case the random variable would be insurers capital X.

$E[U_n(X)/U_0(X)] = E[Ln(X_n/X_0)]$

Mathematically, over a n year period, if the insurer commits a fixed fraction f of its total current capital each period according to $f X_{i-1}$, where $0 < f <= 1$, with Outcome O_i each period then Capital at the end of n periods X_n can be written as

$$Xn = X_0 + \sum_{i=1}^{n} \left(O_i f X_{i-1} \right)$$

If S and F are the number of periods with no loss and full limit losses, respectively, in n periods, then insurers capital after n periods can be defined as

$X_n = X_0(1+f)^S(1-f)^F$, where $S+F = n$. With f in the interval $0 < f < 1$.

The exponential rate of increase per period then can be defined as

$$G_n(f) = Ln\,[X_n/X_0]^{(1/n)} = S/n\,Ln(1+f) + F/n\,Ln(1-f)$$

Let $\frac{S}{n} = p,\ probability\ of\ no\ loss$ and

$\frac{F}{n} = q,\ probability\ of\ a\ full\ limit\ loss$

Kelly chose to maximize the expected value of the growth rate coefficient, $g(f)$, defined as

$$g(f) = E\ [Ln\,(X_n/X_0)^{(1/n)}] = E\,[S/n\,Ln(1+f) + F/n\,Ln(1-f)\,]$$

$$= p\,Ln(1+f) + q\,Ln(1-f).$$

Note that we can also write,

$$g(f) = (1/n)E(Ln\,X_n) - (1/n)Ln\,X_0$$

So for n fixed, maximizing $g(f)$ is the same as maximizing $E(Ln\,X_n)$.

Taking derivative with respect to f and equalizing to 0 to maximize the expected value of the logarithmic growth rate function we get

$$g'(f) = p/(1+f) - q/(1-f)$$

$$= (p - q - f)/(1+f)(1-f) = 0 \text{ when } f = f^* = p - q.$$

$$Now\ g''(f) = -p/(1+f)^2 - q/(1-f)^2 < 0$$

So the maximum of $g(f)$, the expected value of logarithmic Capital growth rate in this case is defined at $f^* = p - q$

In summary, this paper presents a systematic economic approach to optimizing capital allocation by leveraging the Kelly criterion. By framing investment decisions within the context of maximizing the expected logarithmic utility of capital, the analysis demonstrates how an investor can balance the trade-off between risk and growth. The derivation of the optimal betting fraction, $f^* = p - q$, illustrates a clear, quantitative strategy to maximize the long-term growth rate of capital while mitigating the risk of ruin. This framework not only reinforces the theoretical foundations of risk-adjusted returns but also provides practical insights for investors aiming to allocate their capital effectively in uncertain markets. Ultimately, the methodology underscores the importance of a disciplined, mathematically grounded approach to investment, paving the way for further research and potential adaptations in more complex, real-world financial environments.